RADIATIVE-CONDUCTIVE HEAT TRANSMISSION THROUGH A MEDIUM WITH A CYLINDRICAL GEOMETRY: PART II

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A solution to the linearized equation of radiative–conductive heat transmission through a cylindrical layer has been found by a numerical method.

In [1] the author has set up and then approximately solved integral equations describing the steady-state field of a semitranslucent medium contained between coaxial cylinders. The physical model for this problem was constructed on the basis of heat transmission by conduction and radiation, taking into account a selectivity of the optical characteristics and a mirror reflection of the radiant heat at the boundary surfaces.

Transformation of the Fundamental Equations. The commonly prevailing condition that $\vartheta(r) \ll T_1$ leads to a linearized equation of the temperature field: Eq. (5) in [1]. We will rewrite this equation as follows:

$$\vartheta(r) = \frac{Qr_1}{\lambda}\ln\frac{r}{r_1} - \frac{4}{\lambda}\Delta T\int\limits_{\nu=0}^{\infty}\varepsilon_{2\nu}n_\nu^2\left(\frac{\partial I_{\text{B}}}{\partial T}\right)_{T_1}[F_1(r)+F_2(r)]\,d\nu$$
$$-\frac{4}{\lambda}\int\limits_{\rho=r_1}^{\rho=r_2}\vartheta(\rho)\left[\int\limits_{\nu=0}^{\infty}k_\nu n_\nu^2\left(\frac{\partial I_{\text{B}}}{\partial T}\right)_{T_1}G(\nu,\ r,\ \rho)\,d\nu\right]d\rho, \quad (1)$$

where

$$G(\nu,\ r,\ \rho) = \begin{cases}\int\limits_{t=\rho}^{r}\left[\int\limits_{z=0}^{r_1/t}f_1(z,\ t)\,dz - \int\limits_{z=r_1/t}^{\rho/t}f_2(z,\ t)\,dz\right]dt + \int\limits_{t=r_1}^{\rho}\left[\int\limits_{z=0}^{r_1/t}f_3(z,\ t)\,dz\right. \\ \left.\qquad + \int\limits_{z=r_1/t}^{1}f_4(z,\ t)\,dz\right]dt; \qquad \rho < r, \\ \int\limits_{t=r_1}^{r}\left[\int\limits_{z=0}^{r_1/t}f_3(z,\ t)\,dz + \int\limits_{z=r_1/t}^{1}f_4(z,\ t)\,dz\right]dt; \qquad \rho > r;\end{cases} \quad (2)$$

$$f_1(z,\ t) = \int\limits_{\psi=0}^{\pi/2}(\{R_{2\nu}\exp[-(2v_2-v_3-v_4)/\cos\psi](1-R_{1\nu}\exp[-2(v_3$$
$$-v_1)/\cos\psi]) - R_{1\nu}\exp[-(v_3+v_4-2v_1)/\cos\psi](1-R_{2\nu}\exp[-2(v_2$$
$$-v_3)/\cos\psi])\}\{1-R_{1\nu}R_{2\nu}\exp[-2(v_2-v_3)/\cos\psi]\}^{-1} - \exp[-(v_3-v_4)/\cos\psi])\,\frac{\cos\psi k_\nu\rho}{v_4}\,d\psi; \quad (3)$$

$$f_2(z,\ t) = \int\limits_{\psi=0}^{\pi/2}(\exp[-(v_3-v_4)/\cos\psi] + \exp[-(v_3+v_4)/\cos\psi]$$
$$+R_{2\nu}\{\exp[-(v_3+2v_2-v_4)/\cos\psi] + \exp[-(v_3+2v_2+v_4)/\cos\psi]$$
$$-\exp[-(2v_2-v_3-v_4)/\cos\psi] - \exp[-(2v_2-v_3$$
$$+v_4)/\cos\psi]\}\{1-R_{2\nu}\exp(-2v_2/\cos\psi)\}^{-1}\,\frac{\cos\psi k_\nu\rho}{v_4}\,d\psi; \quad (4)$$

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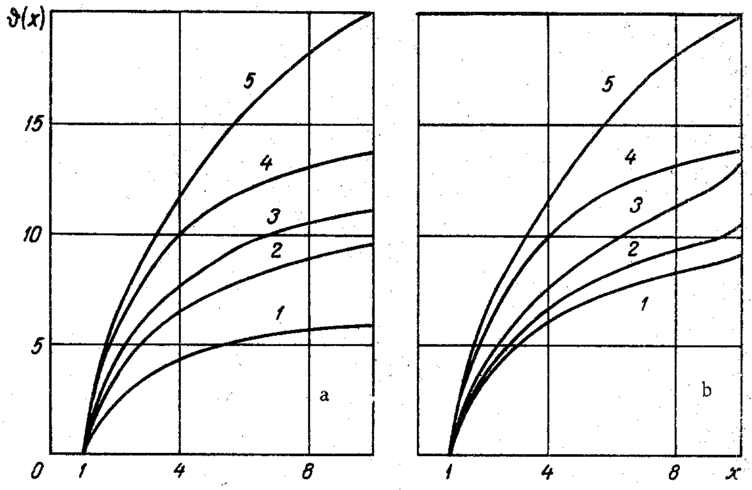

Fig. 1. Temperature profile of a cylindrical layer $\vartheta(x) = T_1 - T(x)$ with $x = \tau/\tau_1$, $\tau_1 = 0.02$, $\tau_2 = 0.2$, N = 57,944; (a) 1) $R_1 = R_2 = 0$; 2) $R_1 = R_2 = 0.5$; 3) $R_1 = 0.8$ and $R_2 = 0.2$; 4) $R_1 = 1.0$ and $R_2 = 0.2$; 5) with radiation disregarded; (b) 1) $R_1 = 0.2$, $R_2 = 0.6$; 2) $R_1 = 0.2$, $R_2 = 0.8$; 3) $R_1 = 0.2$, $R_2 = 1.0$; 4) $R_1 = 1.0$, $R_2 = 0.2$; 5) with radiation disregarded.

$$f_3(z,\ t) = \int\limits_{\psi=0}^{\pi/2} \Big(\{R_{2\nu} \exp[-(2v_2 - v_3 - v_4)/\cos\psi]\,(1 - R_{1\nu} \exp[-2(v_3 - v_1)/\cos\psi]) - R_{1\nu} \exp[-(v_3 + v_4 - 2v_1)/\cos\psi]\,(1 - R_{2\nu} \exp[-2(v_2 - v_3)/\cos\psi])\} \times \{1 - R_{1\nu}R_{2\nu} \exp[-2(v_2 - v_3)/\cos\psi]\}^{-1} + \exp[-(v_4 - v_3)/\cos\psi]\Big) \frac{\cos\psi k_\nu \rho}{v_4}\, d\psi; \tag{5}$$

$$f_4(z,\ t) = \int\limits_{\psi=0}^{\pi/2} \Big(\exp[-(v_4 - v_3)/\cos\psi] - \exp[-(v_3 + v_4)/\cos\psi] - R_{2\nu}\{\exp[-(2v_2 + v_3 - v_4)/\cos\psi] + \exp[-(2v_2 + v_3 + v_4)/\cos\psi] - \exp[-(2v_2 - v_3 - v_4)/\cos\psi] - \exp[-(2v_2 - v_3 + v_4)/\cos\psi]\}\{1 - R_{2\nu} \exp[-2v_2/\cos\psi]\}^{-1}\Big) \frac{\cos\psi k_\nu \rho}{v_4}\, d\psi; \tag{6}$$

$v_1 = k_\nu \sqrt{r_1^2 - t^2z^2}$; $v_2 = k_\nu \sqrt{r_2^2 - t^2z^2}$; $v_3 = k_\nu t\sqrt{1 - z^2}$; $v_4 = k_\nu \sqrt{\rho^2 - t^2z^2}$, and functions $F_1(r)$, $F_2(r)$ are defined by the triple integrals in [1]. Changing the variables

$$x = \frac{r}{r_1};\quad y = \frac{\rho}{r_1};\quad w = \frac{t}{r_1}$$

and introducing the notation $k_\nu r = \tau$, $k_\nu \rho = \tau'$, $k_\nu r_1 = \tau_1$, $\tau_2/\tau_1 = m$, instead of (1) and (2) we find

$$\vartheta(x) = \frac{Qr_1}{\lambda} \ln x - \frac{4}{\lambda}\,\vartheta(m) \int\limits_{\nu=0}^{\infty} \frac{\varepsilon_{2\nu} n_\nu^2}{k_\nu} \left(\frac{\partial I_{\text{B}}}{\partial T}\right)_{T_1} [F_1(x) + F_2(x)] - \frac{4r_1}{\lambda} \int\limits_{y=1}^{m} \vartheta(y) \left[\int\limits_{\nu=0}^{\infty} k_\nu n_\nu^2 \left(\frac{\partial I_{\text{B}}}{\partial T}\right)_{T_1} G(\nu,\ x,\ y)\, d\nu\right] dy; \tag{7}$$

$$G(\nu,\ x,\ y) = \begin{cases} r_1 \int\limits_{w=y}^{x} \left[\int\limits_{z=0}^{1/w} f_1(z,\ w)\, dz - \int\limits_{z=1/w}^{y/w} f_2(z,\ w)\, dz\right] dw \\ \quad + r_1 \int\limits_{w=1}^{y} \left[\int\limits_{z=0}^{1/w} f_3(z,\ w)\, dz + \int\limits_{z=1/w}^{1} f_4(z,\ w)\, dz\right] dw; & y < x; \\ r_1 \int\limits_{w=1}^{x} \left[\int\limits_{z=0}^{1/w} f_3(z,\ w)\, dz + \int\limits_{z=1/w}^{1} f_4(z,\ w)\, dz\right] dw; & y > x. \end{cases} \tag{8}$$

TABLE 1. Comparison between Exact Solution and Approximate Solution ($\tau_1 = 0.02$, $\tau_2 = 0.2$, N = 57.944, $\Delta T_0 = 20°C$)

R_1	R_2	ΔT	$\Delta T^{(1)}$	$\delta = \frac{\Delta T - \Delta T^{(1)}}{\Delta T}$, %
0	0	5,98	5,18	13,4
0,5	0,5	9,60	8,33	12,7
0,2	0,6	8,96	8,57	4,5
0,2	0,8	10,4	11,2	—8,1
0,8	0,2	11,0	8,0	26,9

The resulting equation, which describes the temperature field of a cylindrical layer of a substance with selective optical characteristics, contains many parameters. Their number could be reduced substantially, if the "gray" approximation were applicable and functions $k(\nu)$, $n(\nu)$ were replaceable by their mean spectral values. It has been shown earlier [2] that such a simplification is permissible for several semitranslucent substances. Letting

$$\frac{Qr_1}{\lambda}\ln\frac{r_2}{r_1} = \Delta T_0;\quad \frac{\vartheta(x)}{\Delta T_0} = \varphi(x);\quad \frac{\ln x}{\ln m} = f(x);\quad N = \frac{16n^2\sigma T_1^3}{\pi k\lambda}, \tag{9}$$

we have for the "gray" approximation

$$\varphi(x) = f(x) - N\varepsilon_2\varphi(m)\,[F_1(x) + F_2(x)] - N\tau_1^2\int\limits_{y=1}^{m}\varphi(y)\,G(x,\,y)\,dy. \tag{10}$$

In this case, evidently, the referred temperature field depends on the following parameters: optical thickness of the inner surface τ_1, ratio $m = \tau_2/\tau_1$, radius of the inner surface and radius of the outer surface, reflectivities R_1 and R_2 of the two surfaces, and parameter N which characterizes the ratio of the two modes of heat transmission.

Features of the Computation Process. The linear integral equations (7) and (10) can be solved only by a numerical method. Most worthwhile here is the use of the quadratures method, which reduces the problem to the solution of a system of linear algebraic equations. Since an unknown value of the sought function at the boundary x = m appears in the equations explicitly, hence it becomes necessary to use closed quadrature formulas here. The Markov formula [3] was actually used. When applied to the integral equations (10), for instance, it yields

$$\int\limits_{y=1}^{m}\varphi(y)\,G(x,\,y)\,dy = \frac{m-1}{2}\sum_{j=1}^{p}A_j\varphi_jG_{ij},$$

where A_j denote the quadrature coefficients, $\varphi_j = \varphi(y_j)$, $G_{ij} = G(x, y_j)$, and points x_i, y_j correspond to the nodes of order p in the Markov formula. Considering that $\varphi(m) = \varphi_p$, we obtain instead of (10) the following system of equations:

$$\sum_{j=1}^{p}\left\{\delta_{ij} + N\varepsilon_2\,[F_1(x_i) + F_2(x_i)]\,\delta_{jp} + \frac{m-1}{2}N\tau_1^2A_jG_{ij}\right\}\varphi_j = f(x_i); \tag{11}$$

$$i = 1,\ 2,\ \ldots,\ p.$$

A basic difficulty in solving this system has to do with calculating the elements of the G_{ij} matrix. Relations (8) and (3)-(6) indicate that G(x, y) is a continuous function in x, y on the interval [1, m]. The triple integrals constituting the elements G_{ij}, however, are in a practical manner evaluated by successive integration with respect to each of the three variables ψ, z, w and, therefore, it becomes necessary to repeatedly evaluate the functions (3)-(6), which at certain values of z, w, y result in improper integrals with respect to variable z. The integrand functions tend to infinity, because the radical $v_4 = \tau_1\sqrt{y^2 - w^2z^2}$ in the denominator vanishes at the edges of the integration interval: at z = y/w for all values of y, w and at z = 1 when w = y. Taking into account that in an improper integral of the kind

$$\int\limits_a^1\frac{f(t)\,dt}{\sqrt{1-t^2}} = \int\limits_a^{1-\Delta}\frac{f(t)\,dt}{\sqrt{1-t^2}} + f(1)\int\limits_{1-\Delta}^{1}\frac{dt}{\sqrt{1-t^2}} = \int\limits_a^{1-\Delta}\frac{f(t)\,dt}{\sqrt{1-t^2}} + f(1)\left[\frac{\pi}{2} - \arcsin(1-\Delta)\right]$$

(here f(t) is a continuous bounded function) the last term becomes arbitrarily small as Δ decreases, inasmuch as $\sin^{-1}t$ is continuous, we conclude that such an integral can be calculated within any desirable accuracy by standard methods, only with the upper limit of integration replaced by an appropriately small quantity Δ. In the process of integration with respect to z, for determining the elements of G_{ij}, we have replaced the limits y/w and 1 by $y/w-\Delta$ and $1-\Delta$ respectively, whereupon repeated computations with different values of Δ have established that $\Delta = 5 \cdot 10^{-3}$ ensures the necessary accuracy.

All calculations were made on a model BÉSM-4 computer. Multiple numerical evaluations of the integrals, each within an accuracy ensuring an immunity of the final results to cumulative errors, are found to be uneconomical in terms of machine time. Thus, for determining the temperature profile of a layer on the basis of 13 points, a solution of Eq. (10) takes about two hours.

Results of the Solution. In Fig. 1a are shown temperature profiles of a cylindrical layer with various combinations of reflectivities at the boundaries. The parameter values for these computations were selected so as to yield a temperature drop of 20°C in the absence of any radiation (curve 5). As is to be expected, the temperature gradient becomes minimum when $R_1 = R_2 = 0$. As the reflectivities increase (curve 2), the gradients at all points increase too. A comparison between curves 3 and 4 in Fig. 1a indicates how the temperature profile changes depending on R_1. If R_1 = const and R_2 is varied, however, then the temperature profile changes according to curves 1, 2, 3 in Fig. 1b. As can be seen, the temperature profile acquires an inflection point and, as R_2 increases, this inflection point becomes more pronounced. The temperature gradients at the "cold" boundary of a layer increase at the same time. It is interesting to compare curves 3 and 4 in Fig. 1b, which correspond to the same pair of R_1, R_2 values differently distributed. It is quite evident that the gradients at the surface increase with increasing reflectivities. This has to do with a declining role of heat radiation in the vicinity of a boundary whose reflectivity is high (and whose emissivity is thus low).

For a comparison with the approximate representation of a temperature field by the free function in Eqs. (7) and (10), which was used in the first part of this article, we have tabulated the values of temperature drops based on the exact solution (ΔT) and on the approximate solution ($\Delta T^{(1)}$) as well as the relative error of the latter. Evidently, the error is most significant when radiation from a surface contributes least to the ambient radiation from the medium. The refinement based on a complete solution of the integral equations is substantial and, therefore, reliable quantitative data concerning the temperature field in the case of strong radiation can only be obtained by a solution of the problem in rigorous form on the basis of Eq. (7) or Eq. (10).

The results obtained here may be useful for studying the thermophysical properties of semitranslucent solid substances at high temperatures, when cylindrical specimens with internal heat sources are often used, as well as for studying the thermal conductivity and the thermal diffusivity of confined liquid specimens by the hot-wire or by the line-source method. It can be stated here that heat radiation plays a lesser role in a cylindrical layer than in a plane layer. The decrease in photon conduction becomes more significant as both the reflectivity of the inner surface and the difference between R_1 and R_2 increases.

NOTATION

Q	is the total energy flux through a layer;
λ	is the thermal conductivity of the substance;
k	is the absorption coefficient;
n	is the refractive index;
ε	is the emissivity of a boundary surface;
R	is the reflectivity of a boundary surface;
ΔT	is the total temperature drop across a layer;
$\vartheta(r) := T_1 - T(r)$;	
$I_B(\nu, T)$	are the Planck functions;
σ	is the Stefan constant;
F_1, F_2	are the functions defined in [1];
r, ρ	are the cylindrical coordinates;
x, y, w	are the dimensionless cylindrical coordinates;
τ	is the optical thickness;
$m = \tau_2/\tau_1$; δ	are the Kronecker deltas;
A_j	are the coefficients in the Markov quadrature formula;
N	is the heat transfer parameter (equality (9)).

Subscripts

1 denotes the inner surface;
2 denotes the outer surface;
ν denotes the spectral values;
0 denotes the absence of radiation.

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